A RESEARCH ON THE DYNAMICS STABILITY OF A FREE-FREE BEAM UNDER THRUST ACCORDING TO TIMOSHENKO BEAM THEORY

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The Slender rockets and missiles are always idealized as free-free beams under a thrust force to investigate the dynamic stability characteristics and the natural frequencies and mode shapes of transverse vibration, those works are usually carried out based on Euler-Bernoulli beam theory. This paper focuses on such projects with low slenderness and establishes the dynamic model of the free-free beam subjected to the effect of thrust on tail based on Timoshenko beam theory and the extended Hamilton's principle. The dimensionless forms of dynamic governing equations with improved boundary conditions are analyzed with the differential quadrature method (DQM). Meanwhile, the corresponding model of Raleigh beam, shear beam and Euler-Bernoulli beam are established according to the degeneration of original equations. Based on the investigation to the effect of different boundary conditions and different beam models on the critical thrust force of instability state, the variations of dynamic stability and the vibration characteristics of the beam model with the thrust force are obtained. The work is expected to contribute important reference for the analyzing of dynamic stability and the design of control system of aircrafts with high elasticity and low mass.

1. Introduction

Missiles and rockets are always with high thrust-to-weight ratio and slender shapes, the significant flexibility leads to dynamic instability appears as a result of the interaction of fluid and structure subjected to end thrust as follower force. Investigations to the dynamic stability of slender flight vehicles can provide important reference for the structure design and navigation control.

It is generally believed that considering the effect of end thrust for slender missiles and rockets is a typical non-conservative issues [1].Such aircrafts obtained thrust from the jet engine, within the control system, the velocity of gas from jet is so high that the direction of thrust follows the vertical direction to the transverse vibration deformation of beam, meanwhile, the variation of cross section area at head of aircraft causes drag force. The effects of thrust at tail and drag at head lead to axial compressive load for the aircraft. For the slender aircrafts idealized as free-free beam model, Beal [2] investigate the stability of a flexible missile idealized as a uniform free beam under an end thrust at the first time, and incorporated a simplified control system to obtain the directional stability. They found that critical thrust magnitude is associated with coalescence of the two lowest bending frequencies, instabilities are most likely to occur when the frequency of the thrust variation is twice any of the bending frequencies. Wu [3] used the finite-element technique to study the stability behavior of a flexible free-free beam with concentrated mass under a constant thrust and subjected to a directional control device. Numerical results of their research show the nonzero rigid motion mode, lack of control to this mode would lead to divergent. Pourtakdoust and Assadian [4] combined the rigid motion equations and governing equations of non-uniform beam subject to axial compressive load, and investigated the effect of thrust on dynamic stability and rigid motion of flexible guided missile. Xu et al. [5] simplified missile subjected to end thrust as non-uniform beam model under follower load to investigate aeroelastic dynamic stability of slender missiles.

All the above model are based on the Euler-Bernoulli beam theory. Ignoring the effect of shear deformation and moment of inertia of cross section leads to reduce the accuracy of the results of these models. Peter et al. [6] studied the dynamic stability of a cantilevered beam lying on an elastic foundation and subjected to a follower load based on Timoshenko beam theory. By Analyzing influence of load factor, concentrated mass and elastic foundation on natural frequencies of beam, it was shown that tangential follower load cause flutter while the Timoshenko beam reaches unstable point, and the amplitude becomes large which cause serious aerodynamic flutter for aircraft, and should be avoided in the structure design. Amir et al. [7] also established multi-stepped beam models with concentrated mass subjected to a follower force based on Timoshenko beam theory, and introduced a parametric concept for analyzing a multi-step aerospace structure. Finite element method has been used to study the dynamic and static instability characteristics, which is limited in of formation of stiffness matrix and numerical development of code. Such analysis is always simplified as eigenvalue solving problems, it is needed to decouple the governing equations while using finite element equation, and the assumption modes shape functions are always hard to satisfy the boundary conditions while using Garlerkin method, compared to these two methods, differential quadrature method (DQM) has littler limitation and is suitable for solving problem of this kind.

The transverse vibration model of free-free beam subjected to thrust force is established with in the Timoshenko beam theory according to extended Hamilton's principle. The dimensionless forms of dynamic governing equations are obtained with improved boundary conditions. Numerical solution of the dimensionless natural frequencies is studied by using DQM, The aims are to compare the effect of different beam models and boundary conditions on the critical thrust of dynamic instability, and the influence on the dynamic characteristics of the beam models.

2. Governing equations and boundary conditions

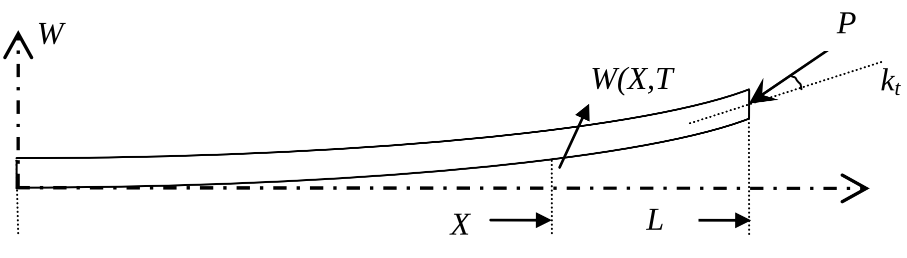

Figure 1. The model of free-free beam subjected to end thrust

Considering an uniform Timoshenko free-free beams subject to end thrust P, and with the transverse displacement $W(X,T)$. EI is the stiffness and $I_{m(x)}$ is the moment of inertia of the beam cross section. The shear stiffness is κGA, κ is the shear ratio of cross section while A is the area of the beam cross section and G is the shear modulus. The length of beam is L and the linear density is ρA, the model of beam is shown in Fig. 1.

The kinetic and potential energies of beam can be expressed as follows:

$$T_B = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial W}{\partial T}\right)^2 dX + \frac{1}{2}\int_0^L \rho A\left(\frac{\partial U}{\partial T}\right)^2 dX + \frac{1}{2}\int_0^L I_m\left(\frac{\partial \Theta}{\partial T}\right)^2 dX \tag{1}$$

$$U_E = \frac{1}{2}\int_0^l EA\left(\frac{\partial U}{\partial X}\right)^2 dX + \frac{1}{2}\int_0^L EI\left(\frac{\partial \Theta}{\partial X}\right)^2 dX + \frac{1}{2}\int_0^L \kappa GA\left(\Theta - \frac{\partial W}{\partial X}\right)^2 dX \tag{2}$$

where $\Theta_{(X,T)}$ denotes the rotary angle of beam cross section. For Timoshenko beams, $\Theta = \partial W/\partial X + \Psi$, where Ψ is the shear angle.

The potential energies caused by external conservative force N is zero, $U_N = 0$.The external conservative work done by axial force $N_{(X)}$ is described as follows:

$$W_F = \frac{1}{2}\int_0^L N_{(X)}\left(\frac{\partial W}{\partial X}\right)^2 dX \tag{3}$$

According to extended Hamilton's principle for the non-conservative system, it can be expressed as:

$$\delta\int_{T_1}^{T_2}\left(T_B - U_E - U_N\right)dT + \int_{T_1}^{T_2}\delta W_F dT = 0 \tag{4}$$

For uniform beams, the moment of inertia $I_{m(x)} = \rho I$. Without considering the coupling of longitudinal and transverse vibrations, the governing equations of transverse vibration can be expressed as:

$$\begin{aligned} &-\rho A\left(\frac{\partial^2 W}{\partial T^2}\right) - \left[\frac{\partial N}{\partial X}\frac{\partial W}{\partial X} + N_{(X)}\frac{\partial^2 W}{\partial X^2}\right] - \frac{\partial}{\partial X}\left[\kappa GA\left(\Theta - \frac{\partial W}{\partial X}\right)\right] = 0 \\ &\kappa GA\left(\Theta - \frac{\partial W}{\partial X}\right) - \frac{\partial}{\partial X}\left(EI\frac{\partial \Theta}{\partial X}\right) - \frac{\partial}{\partial T}\left(I_m\frac{\partial \Theta}{\partial T}\right) = 0 \end{aligned} \tag{5}$$

For slender aircrafts idealized as beam models, the axial force at X position $N_{(X)}$, and the derivative of axial load N_X are described as:

$$N_{(X)} = P - \frac{P}{\int_0^L \rho A dx}\int_X^L \rho A dx = P\frac{X}{L}; \quad N_X = \frac{P}{L} \tag{6}$$

Substituting Eq. (6) into Eq. (5), the governing equations can be obtained as follows:

$$\begin{aligned} &\frac{\rho I}{\kappa GA}W_{TTTT} + W_{TT} + \frac{P}{\rho AL}W_X + \frac{P}{\rho A}W_{XX} - \frac{\rho I}{\kappa GA}\frac{P}{\rho AL}W_{XTT} + 3\frac{EI}{\kappa GA}\frac{P}{\rho AL}W_{XXX} \\ &-\left(\frac{EI}{\kappa GA} + \frac{I}{A} - \frac{PI}{\kappa GA^2 L}X\right)W_{XXTT} + \left(\frac{EI}{\rho A} - \frac{PEI}{\kappa GA^2 \rho L}X\right)W_{XXXX} = 0 \end{aligned} \tag{7}$$

Define k_t as the directional control factor, shown in Fig.1. The boundary condition of free-free beam is shear force Q and bending moment M at head and tail is all zero, can be expressed as $M_{(X)} = 0, \quad Q_{(X)} = 0; \quad X = 0, L$, one may deduce the following boundary conditions:

$$\begin{cases} \kappa GA\left(\Theta - \dfrac{\partial W}{\partial X}\right) = 0; \quad \dfrac{\partial \Theta}{\partial X} = 0; \quad X = 0 \\ \kappa GA\left(\Theta - \dfrac{\partial W}{\partial X}\right) + Pk_t\dfrac{\partial W}{\partial X} = 0; \quad \dfrac{\partial \Theta}{\partial X} = 0; \quad X = L \end{cases} \tag{8}$$

The following dimensionless parameters can be introduced to simplify the governing equations:

$$x = \frac{X}{L}, \quad w = \frac{W}{L}, \quad t = T\sqrt{\frac{EI}{\rho AL^4}}; \quad k_1 = \frac{I}{AL^2}, \quad k_2 = \frac{E}{\kappa G}, \quad k_3 = \frac{PL^2}{EI} \tag{9}$$

where the dimensionless parameters k_1 represent the effect of rotary inertia, k_2 account for the effect of shear distortion, k_3 represent stiffness of the beam subjected to axial load N, and also account for load factor, respectively.

Eq. (7) can be changed to:

$$\begin{aligned} & k_1^2 k_2 w_{tttt} + w_{tt} + k_1^2 k_3 k_2 w_{xtt} + k_3 w_{xx} + 3k_1 k_2 k_3 w_{xxx} \\ & + \left(k_1 k_2 + k_1 - k_1^2 k_3 k_2 x\right) w_{xxtt} + \left(1 - k_1 k_2 k_3 x\right) w_{xxxx} = 0 \end{aligned} \tag{10}$$

(a) Considering the load directional control, boundary conditions of beam can be expressed:

$$\begin{aligned} & w_{xx} = 0, \quad w_{xxx} = 0; \quad x = 0 \\ & w_{xx} = 0, \quad w_{xxx} + k_3 k_t w_x = 0; \quad x = 1 \end{aligned} \tag{11}$$

(b) Without considering the load directional control factor, the thrust is always normal to the direction of transverse vibration deformation, boundary conditions of beam is the special case of Eq. (11) while k_t=0, obviously.

Meanwhile, the governing equations of Euler-Bernoulli beam can be obtained by ignoring the items associated with the influence of shear deformation and moment of inertia, shown as follows:

$$w_{tt} + k_3 w_{xx} + w_{xxxx} = 0 \tag{12}$$

While only ignoring the effect of shear deformation, Rayleigh beam model can be obtained:

$$w_{tt} + k_3 w_{xx} + k_1 w_{xxtt} + w_{xxxx} = 0 \tag{13}$$

While ignoring the effect of moment of inertia only, the governing equation of shear beam model can be expressed as:

$$\begin{aligned} & k_1^2 k_2 w_{tttt} + w_{tt} + k_1^2 k_3 k_2 w_{xtt} + k_3 w_{xx} + 3k_1 k_2 k_3 w_{xxx} \\ & + \left(k_1 k_2 - k_1^2 k_3 k_2 x\right) w_{xxtt} + \left(1 - k_1 k_2 k_3 x\right) w_{xxxx} = 0 \end{aligned} \tag{14}$$

3. Numerical Solutions and rule of stability

3.1 DQM method

The general solution in the form of separation of variables of Eq. (10) can be expressed as

$$w(x,t) = \phi(x) e^{\lambda t} \tag{15}$$

where ϕ is the mode shape function, λ denotes the natural frequency of beam.

Substituting Eq. (15) into Eq. (10) yields:

$$\begin{aligned} & k_1^2 k_2 \phi \lambda^4 + \left[\phi + k_1^2 k_2 k_3 \phi' - \left(k_1 k_2 + k_1 - k_1^2 k_2 k_3 x\right) \phi''\right] \lambda^2 \\ & + \left[k_3 \phi'' + 3k_1 k_2 k_3 \phi''' + \left(1 - k_1 k_2 k_3 x\right) \phi''''\right] = 0 \end{aligned} \tag{16}$$

Boundary conditions of Eq. (12) can be translated to:

$$\begin{cases} \phi'' = 0, \quad \phi''' = 0, \quad x = 0; \\ \phi'' = 0, \quad \phi''' + k_3 k_t \phi' = 0, \quad x = 1; \end{cases} \tag{17}$$

The nodes are set as [8]:

$$x_1 = 0, \quad x_2 = \delta, \quad x_{n-1} = 1 - \delta, \quad x_n = 1, \quad x_i = \frac{1}{2}\left[1 - \cos\frac{(i-1)\pi}{n-1}\right], \quad i = 3, 4 \ldots, n-2 \tag{18}$$

According to DQM:

$$f_x^{(r)}(x_i) = \sum_{j=1}^{n} A_{ij}^{(r)} f_j; \quad i = 1, 2, \ldots n \tag{19}$$

where $A_{ij}^{(r)}$ is the r^{th} weighted coefficient, and f_j is the value of function at location x_j, n is the number of the nodes.

$$\phi'(x_i)=\sum_{j=1}^{n}A_{ij}^{(1)}\phi_j,\quad \phi''(x_i)=\sum_{j=1}^{n}A_{ij}^{(2)}\phi_j,\quad \phi'''(x_i)=\sum_{j=1}^{n}A_{ij}^{(3)}\phi_j,\quad \phi''''(x_i)=\sum_{j=1}^{n}A_{ij}^{(4)}\phi_j \tag{20}$$

The first order weighting coefficients are [9] :

$$A_{ij}^{(1)}=\begin{cases}\prod_{k=1,k\neq i}^{n}(x_i-x_k)\Big/(x_i-x_j)\prod_{k=1,k\neq j}^{n}(x_j-x_k); & i,j=1,2...n,i\neq j\\ \sum_{k=1,k\neq i}^{n}\dfrac{1}{(x_i-x_k)}; & i,j=1,2...n,i=j\end{cases} \tag{21}$$

The higher order of r^{th} :

$$A_{ij}^{(r)}=\begin{cases}r\left(A_{ii}^{(r-1)}A_{ij}^{(1)}-\dfrac{A_{ij}^{(r-1)}}{(x_i-x_j)}\right); & i,j=1,2...n,i\neq j\\ -\sum_{k=1,k\neq i}^{n}A_{ik}^{(r)}; & i,j=1,2...n,i=j\end{cases} \tag{22}$$

Substituting Eqs. (19)-(22) into Eq. (16) , the governing equation can be translated to the matrix form:

$$\left(\mathbf{B}^{(4)}\lambda^4+\mathbf{B}^{(3)}\lambda^3+\mathbf{B}^{(2)}\lambda^2+\mathbf{B}^{(1)}\lambda+\mathbf{B}^{(0)}\right)\mathbf{\Phi}=\mathbf{0} \tag{23}$$

where $\mathbf{B}_i,\ \ i=0,1,2,3,4$ are $n\times n$ matrix, the mode shape vector $\mathbf{\Phi}=(\phi_j)^{\mathrm{T}}$, $j=1,2...n$.

The solving of Eq. (23) is transformed to the generalized eigenvalue problem. The eigenvalues of Eq. (23) represent the dimensionless natural frequency of beam system. According to the good accuracy of astringency of DQM, set node number n=14 for the numerical analysis.

3.2 Criterion of stability

According to the introduced dimensionless parameters, the dimensionless natural frequencies of beam can be expressed as:

$$\omega_i=\bar{\omega}_i\sqrt{EI/\rho AL^4} \tag{24}$$

where ω_i is the i^{th} dimensionless frequencies, represents the i^{th} extended eigenvalue of Eq. (23). $\bar{\omega}_i$ is the i^{th} original natural frequencies of beam model.

The stability of beam system is determined by the real part $\mathrm{Re}(\omega)$ and imaginary part $\mathrm{Im}(\omega)$ of ω_i. When $\mathrm{Re}(\omega)$ is negative or zero, the structure is considered stable, if $\mathrm{Im}(\omega_i)\neq 0$, it represents dynamic instability, while $\mathrm{Im}(\omega_i)=0$, it represents static instability. When $\mathrm{Re}(\omega)$ is nonzero and positive, this is the case of instability. When $\mathrm{Im}(\omega)$ is zero, divergence instability appears, whereas flutter instability is characterized by $\mathrm{Im}(\omega)\neq 0$. When $\mathrm{Re}(\omega)$ is positive or zero, while $\mathrm{Im}(\omega)$ is zero, the solution represents a rigid body motion.

At least one eigenvalue $\mathrm{Re}(\omega_j)<0,\ \ j=1,2,\cdots n_{\mathrm{mode}}$, the other eigenvlaues $\mathrm{Re}(\omega_i)\leq 0,\ \ i=1,2,\cdots n_{\mathrm{mode}},\ \ i\neq j$, it represents the state of critical stability, thrust P is the critical thrust P_{cr}.

Previously defined dimensionless parameter $k_3=PL^2/EI$ represents the load factor, by introducing the reference critical thrust $P_{cr0}=\pi^2L^2/EI$, the load factor $k_3=\pi^2\,P/P_{cr0}$.

4. Numerical results and discussion

Set a typical simplified slender aircraft with rectangular cross-section model as uniform beam, L=8m, I=0.0119m^4, E=100GPa, A=0.181m^2, Poisson's ratio is 0.3, ρ =1.986×10^3, to investigate the dynamic stability of beam with the influence of different factors.

4.1 Stability analysis of different boundary type and different beam model

Thrust direction control k_t=0 means that the directional control of thrust is ignored. The first three order dimensionless frequencies of different beam models are compared in Fig. 2. The result is similar to Beal[3], Wu[4] and Siamak[10] , it shows that before k_3=40, the first three order dimensionless frequencies decrease with the increasing thrust, after that, the first order frequencies turns to increase and couples with the second order at k_3=80, the thrust reaches the critical value and beam is dynamic unstable, the third order frequency remains to decrease.

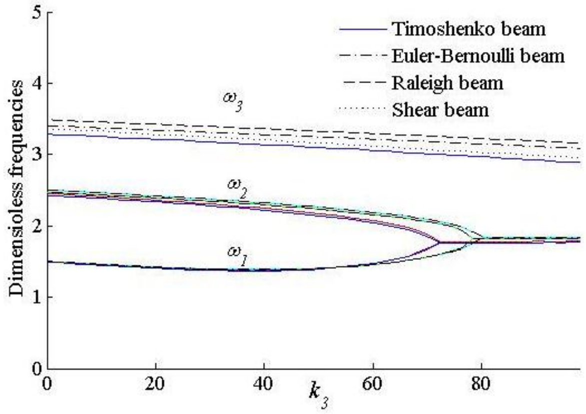

Figure 2. First three dimensionless frequencies of different beam model VS. thrust factor, k_t=0

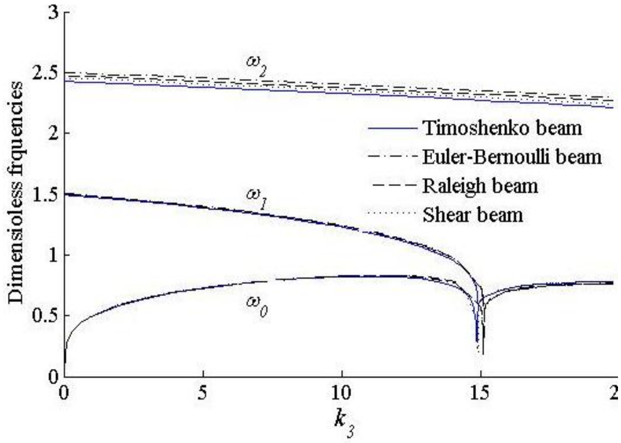

Figure 3. Foundational and first two dimensionless frequencies of different beam model VS. thrust factor, k_t=1

Comparative results of different beam models show that the dimensionless frequencies of Timoshenko beam is the lowest than other beam models for every order, those of Euler-Bernoulli beam are highest, the difference is higher for high order frequencies than low orders, which meet the dynamic characteristics of pinned-pinned beam [11]. The critical thrust factor for Timoshenko beam, Euler-Bernoulli beam, Rayleigh beam and shear beam model are 72.3, 80.9, 78.7 and 72.5, respectively. It can be concluded that the effect of shear deformation and moment of inertia both reduce the critical thrust, and the influence of moment of inertia plays the main role.

When directional control factor of thrust k_t=1, the dimensionless frequencies of different beam models are shown in Fig. 3. The result is different from the case of k_t=0. Firstly, unstable rigid mode appears while ω_0 represents its dimensionless frequencies. The frequency of rigid mode does not remains zero and increases with the thrust, Secondly, the 1st and 2nd order frequencies of elastic bending mode decline. When k_3 reaches to 15, the 1st order frequency decreases to zero as well as the rigid mode. After that the new coupled elastic mode appears and increases with thrust. The value of thrust factor at first couple point for Timoshenko beam is 14.8 while it is the 15.2 for Euler-Bernoulli beam, highest than other beam models.

It also should be noted that the first order dimensionless frequency of beam for different boundary conditions is always 1.5, which is the same as free-free beam without the effect of thrust and axial load.

4.2 Relationship between thrust directional control factor and dynamic stability

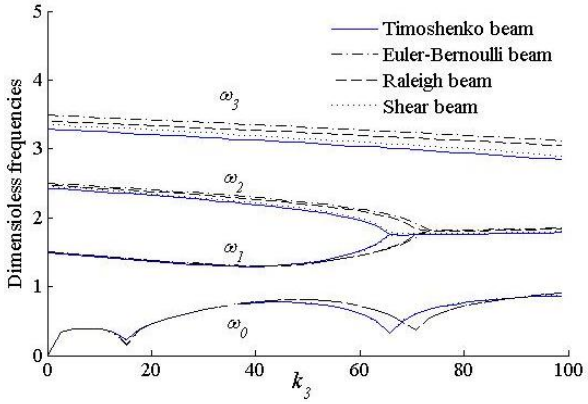

Figure 4. Foundational and first three dimensionless frequencies of different beam model VS. thrust factor, k_t=0.1

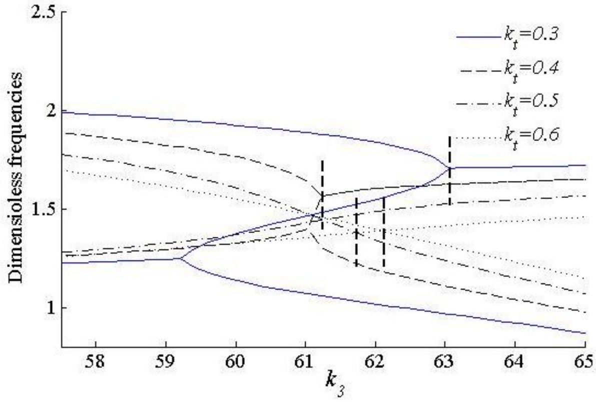

Figure 5. Variation of dimensionless frequencies with different directional control factor of thrust

When the thrust direction control factor k_t=0.1, the stability of beam appears to be different from the above two cases, as shown in Fig. 4, but it also contains both nonzero rigid modes and the coupling of 1st and 2nd elastic bending mode. The dimensionless frequency of the rigid mode varies with the increasing thrust. The critical thrust factor k_3 declines to near 70 for different beam models.

From the above results it can be concluded that the thrust directional control factor affects the dynamic stability, to investigate the law of their relationship, Euler-Bernoulli beam model is used to analyzed the critical thrust factor, in order to avoid the influence of shear deformation and moment of inertia effect, the result is shown in Fig. 5.

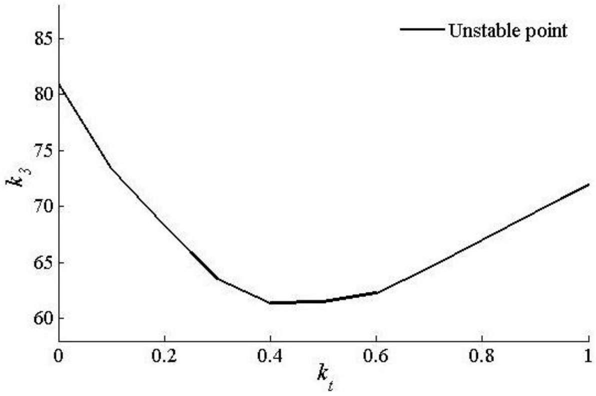

Figure 6. Variation of unstable point with different directional control factor of thrust

When k_t=0.3, the 1st order frequency bifurcates while thrust factor reaches to 59.5, and couples with the 2nd frequency at 63.3. When k_t is larger, reaches to 0.4 and 0.5, the divergence point declines and coincides with the coupling point, the beam undergoes dynamic instability.

Further analysis of coupling point varying with the thrust factor is shown in Fig. 6. When thrust factor become larger than k_t=0, the coupling points firstly decrease and reaches the minimize value at k_t=0.4~0.5, then it increases and reaches another maximum point of value 73 when k_t=1, which is lower than k_t=0.

5. Conclusion

In the present paper, the transverse vibration governing equations of free-free Timoshenko beam subjected to end thrust is established according to Timoshenko beam theory, and DQM method is used in order to investigate the dynamic stability characteristics of beam numerically, results show that:

1. The effect of the shear deformation and moment of inertia both declines the critical thrust factor of beam model, and moment of inertia plays the main role.
2. As the increasing of thrust, the 1st and 2nd elastic bending modes couple when the directional control factor of thrust is low, the beam undergoes dynamic instability. The rigid modes appears as when the directional control factor of thrust is near to 1, while elastic modes bifurcates and couple .
3. The directional control factor of thrust has great influence on the critical thrust, the wrong directional control of thrust may lead to reduce the critical thrust of the aircraft.

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