AN APPROACH OF SOLVING MOVING LOAD PROBLEMS BY ABAQUS AND MATLAB USING NUMERICAL MODES

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Moving load problems have many important applications, like a train passing across a bridge, and have been studied for a long time by numerous researchers. The mathematical and numerical methods to solve these problems are now quite mature but their implementation into practical applications is not easy because of the absence of popular commercial structural analysis software for these specific problems.

This paper presents a general method to solve moving load problems by using a commercial Finite Element (FE) software package (ABAQUS) and self-created MATLAB programs. Modal Superposition (MS) method in analytical form is generally more efficient compared with the FE method. However, analytical modes of complicated structures normally are not available. In this case, the numerical modes of the structures can be obtained by the FE method. In this paper, the approach of using the MS method coded in MATLAB to solve moving load problems with numerical modes from ABAQUS is presented. The idea is that analytical modes of the structures can be approximated by numerical modes used in the MS method. The FE model of the structures can be built in ABAQUS and its numerical modes are easy to obtain and are exported into the MATLAB programs. The equations of motion of vehicles can be derived analytically and coded in MATLAB. An iterative method is adopted and also coded in MATLAB to implement the force equilibrium and contact conditions at the wheel-rail interfaces, thus avoiding updating the mass matrix and stiffness matrix of the whole system in every time step. To verify this approach, the moving mass problem is treated as the benchmark. Numerical results show that this approach produces accurate results.

1. Introduction

Moving load problems are a kind of problems in which the load changes its location in time. Examples are many, like a train crossing a bridge, a bullet being shot out from a gun, a working crane and so on [1, 2]. The interaction between moving vehicles and bridges is an important problem needed to solve accurately for developing safe and economic railways or highways. One main difficulty in solving this problem, taking a railway for example, is that the contact force between a wheel and a rail varies with time and is dependent on the deflection of the rail at contact interface as
well as the displacement of the wheel, which leads to a coupled problem [3]. Another big difficulty is to solve the dynamic contact problem between the rail and the wheel [4]. The contact problem is nonlinear. The contact area and pressure distribution within the contact area are unknown. If local strains or stresses of the rail and the wheel at their contact interface are not important, a point-wise hard contact model could be adopted to simplify the contact problem.

In order to solve the dynamic response of bridges under moving vehicles, Modal Superposition (MS) method can be used to separate space part from time part of the whole solution [5, 6]. The whole solution is equal to the sum of different modes times their corresponding modal coordinates. Another way to solve separately for the whole solution is the Finite Element (FE) method [7]. By using the FE method, the whole solution domain is discretized into a number of elements in space and nodal coordinates in time. The solution at a point within an element equals to the nodal coordinates of the element times shape functions of the element. As only the first several modes are needed to reach engineering accuracy for the whole solution of big structures, the MS method is generally thought to be highly efficient. However, the analytical modes of complicated structures are not available. In this case, their numerical modes can be obtained alternatively by FE method. A good idea is that the analytical modes of complicated structures could be approximated by their numerical counterparts [8].

After applying the MS method, only generalised coordinates in time domain are the remaining unknowns in the equation of motion of the bridge. The next step is to solve the two sets of equations of the bridge and the vehicles. There are two methods available: the coupled method [9] and the iterative method [10]. The coupled method is to eliminate the unknown contact force from the equation of motion of the bridge, leaving deflection of the bridge and some displacements of the vehicles as the only unknowns. The contact force can be replaced by the equation of motion of the vehicles and the displacement compatibility at the corresponding contact interface is imposed. Then the equation of motion of the whole system actually becomes one with time-varying coefficient matrices, which then can be solved by a number of numerical integration methods. On the other hand, the iterative method solves the two sets of equations separately and displacement compatibility and force equilibrium at the contact interface are achieved through an iteration algorithm. A numerical integration method is still needed but the equations do not have time-varying coefficient matrices. It is widely believed that the iterative method is more efficient for large structures, though convergence in the iteration may be an issue.

2. MS method by numerical modes

The contact stiffness between wheels and rails is modelled by discrete springs for the purpose of applying the iterative method. The MS method is adopted to reduce computational time. As the analytical modes of complicated structures are not available, their numerical modes are obtained by an FE software package (ABAQUS) to approximate the analytical ones. The moving sprung mass model in Fig.1 is adopted here to demonstrate the approach of solving the vehicle-bridge interaction problem by using numerical modes in the MS method.

For the moving mass problem shown in Fig. 1, the equations of motion of the beam and the mass can be written as

\[ \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = k_v(u - w(\nu t, t)) \cdot \delta(x - \nu t) \]

\[ -m_v g - k_v(u(t) - w(\nu t, t)) = m_v \frac{d^2 u(t)}{dt^2} \]
where $\rho$ and $E$ are the density and Young’s modulus of the beam respectively; $A$ and $I$ are the area and moment of inertia of beam’s cross section; $k_v$ is the stiffness of the spring between the mass and the beam; $m_v$ is the sprung mass; $w$ and $u$ are the transverse deflection of the beam and the vertical displacement of the mass respectively.

![Moving sprung mass model](image)

**Figure 1.** Moving sprung mass model.

Equation (2) can be rewritten as

$$m_v \frac{d^2 u(t)}{dt^2} + k_v u(t) = -m_v g + k_v \cdot w(vt,t) \tag{3}$$

Applying the MS method to Eq. (1) by multiplying $\int_0^l \varphi_i(x)dx$ to both sides of the equation, then Eq. (1) can be changed to

$$\ddot{q}_i + \omega_i^2 q_i = k_v \left[ u(t) - \sum_{j=1}^n \varphi_j(vt) \cdot q_j(t) \right] \cdot \varphi_i(vt) \tag{4}$$

where $q_i$ is the modal coordinate and $n$ is the number of modes used; $i$ is an integer between 1 and $n$; $\varphi$ is the mode shape of the beam.

Combining $n$ equations of Eq. (3) and Eq. (4) together leads to

$$m_v \ddot{u}(t) + k_v u(t) = -m_v g + k_v \varphi^T q \tag{5}$$

$$M\ddot{q} + Kq = k_v(u(t) - \varphi^T q)\varphi \tag{6}$$

where $q = [q_1, q_2, ..., q_n]^T$, $\varphi = [\varphi_1(vt), \varphi_2(vt), ..., \varphi_n(vt)]^T$, $\omega_i^2 = \frac{EI}{\rho A} \left( \frac{\pi}{l} \right)^4$, $M_{ij} = \delta_{ij}$, $K_{ij} = \omega_i^2 \delta_{ij}$.

$\varphi$ in Eq. (5) and Eq. (6) is the vector of the analytical modes of the beam which can be approximated by their numerical counterparts. The FE model of the beam can be built in ABAQUS and its numerical modes can be obtained by Modal analysis and exported into MATLAB.

### 3. Iterative method

After Eq. (5) and Eq. (6) are derived, the iterative method is implemented in MATLAB to solve the two sets of equations, avoiding updating mass and stiffness matrixes in the equation of motion of the beam. The iterative method used in this paper is referred to the iteration scheme performed in every time step. The Newmark integration method is combined with the iterative method in every time step. The iterative method is demonstrated by the moving sprung mass model in Fig. 1. How-
ever, the application of the iterative scheme demonstrated in this paper can be extended to other complicated vehicle-bridge models.

From time $t_0$ to time $t_0 + \Delta t$, the iterative scheme is working as follows [11]:

Step 1. calculate $0\mathbf{s}_b = \mathbf{M}(a_1^0\mathbf{q} + a_3^0\mathbf{q} + a_4^0\mathbf{q})$ and $0\mathbf{s}_v = m_v(a_1^0\ddot{u} + a_3^0\ddot{u} + a_4^0\ddot{u})$, where $a_1 = \frac{1}{\alpha \Delta t^2}$, $a_3 = \frac{1}{\alpha \Delta t}$, $a_4 = \frac{1}{2\alpha}$ and $\alpha$ is a Newmark integration parameter;

Step 2. assume the initial $\mathbf{q}$ at time $t + \Delta t$ as $\Delta t \mathbf{q}^0 = 0\mathbf{q}$;

Step 3. calculate $\Delta t \mathbf{q}^0 = a_2(\Delta t \mathbf{q}^0 - 0\mathbf{q}) - a_5^0\mathbf{q} - a_6^0\mathbf{q}$ and $\Delta t \mathbf{q}^0 = a_4(\Delta t \mathbf{q}^0 - 0\mathbf{q}) - a_3^0\mathbf{q} - a_4^0\mathbf{q}$, where $a_2 = \frac{\beta}{2\Delta t}$, $a_5 = \frac{\beta}{\alpha} - 1$, $a_6 = (\frac{\beta}{2\alpha} - 1)\Delta t$, $\alpha$ and $\beta$ are Newmark integration parameters;

Step 4. calculate $\Delta t \mathbf{p}^0 = -m_v \Delta t \mathbf{f}^T \Delta t \mathbf{q}^0$;

Step 5. the equation of motion of the mass after Newmark integration can be obtained as

$$(k_v + a_1 m_v)\Delta t u^0 = \Delta t \mathbf{p}^0 + 0\mathbf{s}_v,$$

so $\Delta t u^0 = \frac{\Delta t \mathbf{p}^0 + 0\mathbf{s}_v}{k_v + a_1 m_v}$;

Step 6. calculate $\Delta t \ddot{u}^0 = a_2(\Delta t u^0 - 0\mathbf{u}) - a_5^0 \ddot{u} - a_6^0 \ddot{u}$ and $\Delta t \ddot{u}^0 = a_4(\Delta t u^0 - 0\mathbf{u}) - a_3^0 \ddot{u} - a_4^0 \ddot{u}$;

Step 7. calculate $\Delta t \mathbf{p}_b^0 = k_v(\Delta t u^0 - \Delta t \mathbf{f}^T \Delta t \mathbf{q}^0)\Delta t \mathbf{f}$;

Step 8. the equation of motion of the beam after Newmark integration can be obtained as

$$(K + a_1 \mathbf{M})\Delta t \mathbf{q}' = \Delta t \mathbf{p}_b^0 + 0\mathbf{s}_b,$$

so $\Delta t \mathbf{q}' = (K + a_1 \mathbf{M})^{-1} \cdot (\Delta t \mathbf{p}_b^0 + 0\mathbf{s}_b)$;

Step 9. check if the following convergent criteria is satisfied

$$\frac{\text{Norm}(\Delta t \mathbf{q}' - \Delta t \mathbf{q}^0)}{\text{Norm}(\Delta t \mathbf{q}' - 0\mathbf{q})} \leq \epsilon$$

(7)

where $\epsilon$ is suggested to be between $1.0 \times 10^{-5}$ and $1.0 \times 10^{-8}$ [11].

If Eq.(7) is satisfied, the iterative scheme from time $t$ to $t + \Delta t$ reaches its end and goes back to step 1 for the next time step, otherwise $\Delta t \mathbf{q}^1$ is updated by $\Delta t \mathbf{q}^1 = \Delta t \mathbf{q}^0 + \eta(\Delta t \mathbf{q}' - \Delta t \mathbf{q}^0)$, where $\eta$ is a relaxation coefficient which is between 0 and 1. This scheme is repeated until the mass leaves the beam.

4. Results

The dynamic response of the moving sprung mass model in Fig. 1 is calculated by the MS method using numerical modes and the iterative method. The properties of the system are the same as those in [12]: the length of the beam $L = 30$ m, $A = 7.73$ m$^2$, $I = 7.84$ m$^4$. Young’s modulus of the beam $E = 2.825 \times 10^{10}$ N/m$^2$, the density of the beam $\rho = 5400$ kg/m$^3$, the mass of the moving mass $m = 4.255 \times 10^4$ kg, the stiffness of the spring $k_v = 1.06 \times 10^6$ N/m. The critical speed of the beam is $u_{cr} = \frac{L \omega_1}{\pi} = 868.384$ km/h.

Figure 2 shows the first four mode shapes of the beam obtained in ABAQUS. The dynamic responses of the moving mass system given by Eq. 3 are very close to those in Fig. 9 in [12], which indicates that the approach adopted in this paper works.
To see the effect of the speed of the mass on the dynamic response of the moving mass system, the maximum responses of the beam at mid-span and the mass have been calculated under different speeds of the mass. The dynamic responses of the moving mass system under mass speeds of 108 km/h and 360 km/h are shown in Fig. 4. The time when maximum responses of the beam at mid-span and the mass occur becomes later when the mass speed increases. Figure 5 shows that when the speed of the mass increases from 0 to around 700 km/h the maximum deflection of the beam at
mid-span increase to the peak first and decrease afterwards. The dynamic deflection of the beam at mid-span can be 0.7 times higher than its static deflection when the speed of the mass is 108 km/h. The relationship of dynamic response of the mass against speed of mass is illustrated in Fig. 6. Basically, the maximum displacement of the mass increases to around 1.7 times the static deflection of the beam at mid-span and decrease afterwards when the speed of the mass increase from 0 to around 700 km/h. The maximum dynamic displacement of the mass emerges with the mass moving at 540 km/h. It should be noticed that the above analysis is based on the case that the ratio of the sprung mass over the beam mass is around 0.034 and the stiffness of the spring is $1.06 \times 10^6$ N/m.

Figure 4. Dynamic responses of beam mid-span and mass at speeds of 108 km/h (a, b) and 360 km/h (c, d).
Figure 5. Maximum dynamic response of beam mid-span against mass speed.

Figure 6. Maximum dynamic response of mass against mass speed.
5. Conclusions

This paper presents an approach to solve moving load problems by using numerical modes in Mode Superposition method. Taking the train-bridge interaction problem for example, the FE model of the bridge is built in ABAQUS and its numerical modes are obtained and exported into MATLAB. The equations of motion of the bridge and the train are established analytically and an iterative method implemented in MATLAB is suggested to solve the two sets of equations. This approach is thought to be efficient and easy to implement for complicated moving load models. Its accuracy is verified by the moving mass model and the effect of speed of the mass on the dynamic response of the system is analysed.

REFERENCES